A Criterion for Permanence and Global Stability in the Periodic n-Competing Species System

Svetlin Georgiev Georgiev, Dimiter Petkov Tsvetkov

We consider a periodic system describing competition among n species and propose conditions under which there exists a unique positive periodic solution which attracts all the solutions of positive initial values.

Introduction

We consider the following system wich descriebes competition among n species in a periodic environment

$$(1) \qquad u_i'(t) = u_i(t)\left[b_i(t) - \sum_{j=1}^{n} a_{ij}(t)u_j(t)\right], \quad 1 \leq i \leq n,$$

where a_{ij} and b_i are continuous positive ω-periodic functions of time t, $n \geq 2$. It is of basic interest to find conditions, under which system (1) has positive ω-periodic solution, which is also unique and stable.

In the pioneer work [1] Gopalsamy proposed the following existence condition

$$(2) \qquad \min_t b_i > \sum_{j=1, j\neq i}^{n} \frac{\max_t(a_{ij})}{\min_t(a_{ij})} \max_t b_j.$$

It was shown by Tineo and Alvarez in [2] that, besides the existence of a positive ω-periodic solution, (2) provides also global stability of (1) with respect to all its positive solutions. The global stability property means that any two positive solutions attract each other as $t \longrightarrow \infty$. Obviously, the global stability implies uniqueness. In [2], the following existence condition

$$(3) \qquad b_i(t) > \sum_{j=1, j\neq i}^{n} a_{ij}(t)U_j(t), \quad \forall t, \quad 1 \leq i \leq n,$$

is proposed, where U_i, $1 \leq i \leq n$, is the unique positive ω- periodic solution of the logistic equation

$$x' = x\left[b_i(t) - a_{ii}(t)x\right].$$

In order to prove the global stability, Tineo and Alvarez [2] required another condition

$$\alpha_i a_{ii}(t) > \sum_{j=1, j\neq i}^{n} \alpha_j a_{ji}(t), \quad \forall t, \quad 1 \leq i \leq n, \tag{4}$$

where α_i, $1 \leq i \leq n$, are some positive constants.

It was proved in [2] that (2) implies (3) and (4), and in this way Tineo and Alvarez improved the result of Gopalsamy [1]. Zhao [3] and Zanolin [4] established the following existence condition

$$\int_0^\omega b_i(t)dt > \sum_{j=1, j\neq i}^{n} \int_0^\omega a_{ij}(t)U_j(t)dt, \quad 1 \leq i \leq n, \tag{5}$$

which also provides permanence of (1). The permanence property (see [4]) means the existence of a positive compact set which absorbs every positive solution of (1) as $t \longrightarrow \infty$. (5) is also suggested by Tineo in [5], who in addition used (4) to obtain global stability without a permanence requirement.

In the present paper we use (5) as a condition that provides existence and permanence of system (1). To obtain the stability we will make use of the iterative scheme that comes from the works of Tineo [5] and [8] in which is shown that system (1) generates a sequence of positive ω-periodic functions $U^k = (U_1^k, U_2^k, \ldots, U_n^k)$, defined as follows: $U^0 \equiv 0$ and U_i^{k+1}, $1 \leq i \leq n$, is the unique positive ω-periodic solution of the logistic equation

$$x'(t) = x(t)\left(b_i(t) - \sum_{j=1, j\neq i}^{n} a_{ij}(t)U_j^k(t) - a_{ii}(t)x(t)\right),$$

for which we have $U^{2k} < U^{2k+2} < U^{2k+1} < U^{2k-1}$, $k \geq 1$. Our stability condition requires existence of some positive constants α_i, $1 \leq i \leq n$, and an integer k such that

$$\alpha_i a_{ii}(t)U_i^{2k}(t) - \sum_{j=1, j\neq i}^{n} \alpha_j a_{ij}(t)U_j^{2k-1}(t) > 0, \qquad \forall t, \qquad 1 \leq i \leq n. \tag{6}$$

Notice that the sum in (6) is rowwise in contrast with (4) in which the sum is columnwise.

An example follows in which the conditions (5) and (6) hold while (4) fails. Consider the equations

$$\left| \begin{array}{l} u_1'(t) = u_1(t)\,[(3+\sin t) - (3+\sin t)u_1(t) - (2+\sin t)u_2(t)] \\ u_2'(t) = u_2(t)\,[(3+\cos t) - (2+\cos t)u_1(t) - (3+\cos t)u_2(t)]\,. \end{array} \right.$$

Here $U_1 \equiv U_2 \equiv 1$ and obviously (5) is valid. On the other hand, the numerical calculations show that

$$\begin{array}{rcl} \alpha_1(3+\sin t)U_1^{10}(t) - \alpha_2(2+\sin t)U_2^{9}(t) & > & 1, \\ \alpha_2(3+\cos t)U_2^{10}(t) - \alpha_1(2+\cos t)U_1^{9}(t) & > & 1, \quad \forall t, \end{array}$$

where $\alpha_1 = 20$ and $\alpha_2 = 22$ (here $k = 5$). Therefore (6) is also valid. Nevertheless (4) fails since it implies the existence of positive costants α_1 and α_2 for which

$$\frac{2+\cos t}{3+\sin t} < \frac{c_1}{c_2} < \frac{3+\cos t}{2+\sin t}, \qquad \forall t.$$

The last is a contradiction because

$$\max_t \left[\frac{2+\cos t}{3+\sin t}\right] > \min_t \left[\frac{3+\cos t}{2+\sin t}\right].$$

The reader can find also many other details and basic properties on the periodic n-competing species problem in the papers of Tineo [8], Cushing [9], Battauz and Zanolin [10], Ahmad and Lazer [12, 13] and Redheffer [14, 15]. This namelist does not pretend to be explicit.

The main result

When we say that a real n-vector is positive we mean that the all its components have the same property. t_0 denotes a fixed initial time.

It is not difficult to see that any solution of (1) with positive initial value $u(t_0)$ is defined and remains positive on the whole half-axes $[t_0, \infty)$.

Our main result is

Theorem 1. *Suppose that condition (5) holds and also that there exist positive constants α_i, $1 \le i \le n$, and an integer k such that (6) is fulfilled. Then (1) has exactly one positive ω-periodic solution and any two positive solutions u^1 and u^2 attract each other as $t \longrightarrow \infty$, i.e.*

$$\lim_{t \longrightarrow \infty} (u^1(t) - u^2(t)) = 0. \tag{7}$$

As we note, the existence part of Theorem 1 is proved in [3, 4, 5]. The existence is based on the fact that condition (5) provides permanence of (1). Let us resume these notes in the following theorem in view of their importance.

Theorem 2.([3], [4]) *Suppose that (5) holds. The system (1) is permanent. More precisely, for every positive compact set $\mathcal{K} \subset \mathcal{R}^n$ there exist constants $c > 0$ and $C > 0$ and $t' \geq t_0$ such that $c \leq u_i(t) \leq C$, $t \geq t'$, $1 \leq i \leq n$, whenever $u(t_0) \in \mathcal{K}$.*

Now it remains to show that under condition (6) there exists exactly one positive ω-periodic solution of (1) which attracts all the positive solutions. The main role for this purpose will be played by the results of Tineo (see [5] and [8]) concerning the iterative scheme approach to system (1). The following theorem is extracted from [5] and [8] in a form suitable for us.

Theorem 3. *Suppose that (5) holds and let k be a fixed integer. Then, for every solution of (1) with an initial value $u(t_0) > 0$ there exists t' such that*

$$U_i^{2k}(t) < u_i(t) < U_i^{2k-1}(t), \qquad t \geq t' \qquad 1 \leq i \leq n.$$

The next theorem comes from Bylov [16] and is also extracted in a form suitable for us. It reflects the rowwise approach to the stability of the nonautonomous linear systems. Below we use the vector norm $\|x\| = \max_i |x_i|$.

Theorem 4. *Suppose we are given a differential n-system $z' = B(t)z$ whith a continuous $n \times n$-matrix B. Suppose also that there exists a constant m with*

$$\left(b_{ii}(t) + \sum_{j=1, j\neq i}^{n} |b_{ij}(t)| \right) \leq m, \qquad t \geq t_0, \quad 1 \leq i \leq n,$$

Then the solutions of our system satisfy the estimate

$$\|z(t)\| \leq \|z(t_0)\| e^{m(t-t_0)}, \quad t \geq t_0.$$

Proof of Theorem 1. Define the strip

$$\mathcal{S}_k(t) = \{U_i^{2k}(t) \leq u_i(t) \leq U_i^{2k-1}(t), \quad 1 \leq i \leq n\}.$$

By Theorem 3 we see that any positive solution of (1) goes inside $\mathcal{S}_k$. On the other hand, using induction by k, one can see that $\mathcal{S}_k$ is invariant with respect to the solutions of (1), i.e. if $u(t_0) \in \mathcal{S}_k(t_0)$ then $u(t) \in \mathcal{S}_k(t)$ for all $t \geq t_0$. Now the problem for the stability of the positive solutions of (1) reduces to the problem for the stability of the solutions whose initial values $u(t_0)$ vary in $\mathcal{S}_k(t_0)$. This remarks allow us to suppose, without loss of generality, that $u(t_0) \in \mathcal{S}_k(t_0)$.

From now on the proof is a modification of the reasoning of Tineo and Alvarez [2]. Denote by $u(t;p)$ the solution of (1) with an initial value $u(t_0;p) = p$ and suppose that $p \in \mathcal{S}_k(t_0)$. In acordance with the well-known theory, we are able to verify that the matrix of the partial derivatives $u_p(t)$ is fundamental to the linear system

$$y_i' = \frac{u_i'(t;p)}{u_i(t;p)} y_i - u_i(t;p) \sum_{j=1}^{n} a_{ij}(t) y_j, \quad 1 \le i \le n.$$

After a change of the variables $z_i(t) = y_i(t)/[\alpha_i u_i(t;p)]$, the last becomes

$$z_i' = \sum_{j=1}^{n} b_{ij}(t) z_j \quad where \quad b_{ij}(t) = -\frac{1}{\alpha_i} \alpha_j a_{ij}(t) u_j(t;p). \tag{8}$$

It follows from (6) that

$$\alpha_i a_{ii}(t) U_i^{2k}(t) - \sum_{j=1, j \neq i}^{n} \alpha_j a_{ij}(t) U_j^{2k-1}(t) \ge m, \qquad t \ge t_0, \quad 1 \le i \le n,$$

for some positive constant m. At this point one can see that

$$b_{ii}(t) + \sum_{j=1, j \neq i}^{n} |b_{ij}(t)| \le -m/\alpha, \quad t \ge t_0 \quad 1 \le i \le n,$$

where $\alpha = \max_i \alpha_i$. Now Theorem 4 implies that the solutions of (8) satisfy the estimate

$$\|z(t)\| \le \|z(t_0)\| e^{-\kappa(t-t_0)}, \quad t \ge t_0,$$

where $\kappa = m/\alpha$. Therefore, the fundamental matrix $u_p(t;p)$ satisfies the estimate

$$\|u_p(t;p)\| \le C e^{-\kappa(t-t_0)}, \quad t \ge t_0, \tag{9}$$

where C is constant independent of $p \in \mathcal{S}_k(t_0)$. Suppose that $p, q \in \mathcal{S}_k(t_0)$. Then the representation

$$u(t;p) - u(t;q) = \int_0^1 u_p(t; (1-s)q + sp)(p-q) ds$$

along with (9) implies the inequality

$$\|u(t;p) - u(t;q)\| \le C e^{-\kappa(t-t_0)} \|p - q\|, \quad t \ge t_0,$$

which completes the proof of (7).

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Svetlin Georgiev Georgiev
Dimiter Petkov Tsvetkov
Faculty of Mathematics and Informatics
University of Veliko Tarnovo
5000 Veliko Tarnovo
Bulgaria